

RADON–NIKODYM THEOREMS FOR NONNEGATIVE FORMS, MEASURES AND REPRESENTABLE FUNCTIONALS

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ABSTRACT. The aim of this note is to establish several Radon–Nikodym type theorems appearing in different areas of mathematics, such as the theory of measures, additive set functions, $*$ -algebras and nonnegative Hermitian forms. In each cases our basic approach is the theory of unbounded operators in Hilbert spaces.

1. INTRODUCTION

The notion of absolute continuity appears in plenty of areas of mathematics such as measure theory, theory of $*$ -algebras, positive operators on Hilbert spaces, non-negative Hermitian forms, etc. However, in most cases these definitions (seemingly) completely differ from each other. At any rate, there is a conspicuous and fundamental correspondence: whenever absolute continuity occurs then an appropriate Radon–Nikodym type theorem can be proved on the representability of the absolute continuous object via the dominating one. In the present work we attempt to analyze the deep relation between the various concepts of absolute continuity in the above mentioned cases, namely, from a functional analytic point of view.

Our paper is organized as follows. In Section 2 we state two Radon–Nikodym type theorems for nonnegative Hermitian forms which are defined on a real or complex vector space. The remaining parts of this note contain several applications of these results. In Section 3 we discuss the case of (finitely) additive set functions and prove the corresponding Radon–Nikodym theorem due to Fefferman [2], cf. also Darst and Green [1]. As a novelty we consider set functions defined on a *ring of sets* in contrast to [2] and [1] where only the case of *algebras* were treated. Section 4 is devoted to the well known Radon–Nikodym theorem of the classical measure theory. The method we use here, although it is of functional analytic nature, completely differing from that of Neumann [4]. Finally, in Section 5 we provide two Radon–Nikodym type theorems for positive functionals on a $*$ -algebra. In particular, we extend a corresponding result of Gudder [3] to the case of representable functionals defined on a not necessarily unital $*$ -algebra and we characterize the absolute continuity among pure functionals on a C^* -algebra.

2. RADON–NIKODYM THEOREMS FOR NONNEGATIVE HERMITIAN FORMS

Let \mathcal{D} be a vector space over the real or complex field. Given a nonnegative Hermitian form \mathfrak{s} on \mathcal{D} , we always associate a Hilbert space $\mathfrak{H}_{\mathfrak{s}}$ with \mathfrak{s} by the standard

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method. That is to say, by setting $\mathcal{N}_{\mathfrak{s}} = \{x \in \mathcal{D} \mid s(x, x) = 0\}$, $\mathfrak{H}_{\mathfrak{s}}$ is obtained by completing the pre-Hilbert space $\mathcal{D}/\mathcal{N}_{\mathfrak{s}}$ endowed with the inner product

$$(x + \mathcal{N}_{\mathfrak{s}} \mid y + \mathcal{N}_{\mathfrak{s}})_{\mathfrak{s}} = \mathfrak{s}(x, y), \quad x, y \in \mathcal{D}.$$

In order to simplify the denotations we shall always write x instead of $x + \mathcal{N}_{\mathfrak{s}}$, in the hope that we do not cause any confusion.

If another nonnegative Hermitian form \mathfrak{t} on \mathcal{D} is given, then we say that \mathfrak{s} is absolutely continuous (in other words, closable, cf. [5, 9]) with respect to \mathfrak{t} if

$$(2.1) \quad \mathfrak{t}(x_n, x_n) \rightarrow 0 \quad \text{and} \quad \mathfrak{s}(x_n - x_m, x_n - x_m) \rightarrow 0$$

imply $\mathfrak{s}(x_n, x_n) \rightarrow 0$ for all sequence $(x_n)_{n \in \mathbb{N}}$ of \mathcal{D} . Following the terminology of Gudder [3], a sequence $(x_n)_{n \in \mathbb{N}}$ satisfying (2.1) will be called a $(\mathfrak{t}, \mathfrak{s})$ -sequence.

Our main result in this section is the following Radon–Nikodym-type theorem:

Theorem 2.1. *Let \mathfrak{s} and \mathfrak{t} be nonnegative Hermitian forms on the real or complex vector space \mathcal{D} such that \mathfrak{s} is absolutely continuous with respect to \mathfrak{t} . Then for each $f \in \mathfrak{H}_{\mathfrak{s}}$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of \mathcal{D} such that*

$$(2.2) \quad (x \mid f)_{\mathfrak{s}} = \lim_{n \rightarrow \infty} (x \mid x_n)_{\mathfrak{t}} \quad \text{for all } x \in \mathcal{D}.$$

Moreover, the convergence is uniform on the set $\{x \in \mathcal{D} \mid \mathfrak{s}(x, x) + \mathfrak{t}(x, x) \leq 1\}$.

Proof. Absolute continuity of \mathfrak{s} with respect to \mathfrak{t} means precisely that the canonical embedding operator J of $\mathcal{D} \subseteq \mathfrak{H}_{\mathfrak{t}}$ into $\mathfrak{H}_{\mathfrak{s}}$, defined by

$$(2.3) \quad Jx = x, \quad x \in \mathcal{D},$$

is closable. Hence the domain $\text{dom } J^*$ of its adjoint is dense in $\mathfrak{H}_{\mathfrak{s}}$. Consider a sequence $(g_n)_{n \in \mathbb{N}}$ of $\text{dom } J^*$ such that $g_n \rightarrow f$ in $\mathfrak{H}_{\mathfrak{s}}$, and for any integer n fix $x_n \in \mathcal{D}$ such that $\|J^*g_n - x_n\|_{\mathfrak{t}} < 1/n$. Then for $x \in \mathcal{D}$, $\mathfrak{s}(x, x) + \mathfrak{t}(x, x) \leq 1$ we infer that

$$\begin{aligned} |(x \mid f)_{\mathfrak{s}} - (x \mid x_n)_{\mathfrak{t}}| &\leq |(x \mid f)_{\mathfrak{s}} - (Jx \mid g_n)_{\mathfrak{s}}| + |(x \mid J^*g_n)_{\mathfrak{s}} - (x \mid x_n)_{\mathfrak{t}}| \\ &\leq \|f - g_n\|_{\mathfrak{s}} + \|J^*g_n - x_n\|_{\mathfrak{t}} \rightarrow 0, \end{aligned}$$

as it is claimed. \square

Hereinafter we shall call the form \mathfrak{s} *pseudo-absolutely continuous* with respect to \mathfrak{t} if for any $f \in \mathfrak{H}_{\mathfrak{s}}$ there exists $(x_n)_{n \in \mathbb{N}}$ of \mathcal{D} satisfying (2.2). We can therefore reformulate Theorem 2.1 as follows: absolute continuity implies pseudo-absolute continuity. It is a natural question whether this statement can be reversed. As Example 2.2 below demonstrates, the answer is negative in general.

Before giving a counterexample we briefly recall the notion of singularity: \mathfrak{s} and \mathfrak{t} are called singular with respect to each other if for each nonnegative Hermitian form \mathfrak{w} the properties $\mathfrak{w} \leq \mathfrak{s}$ and $\mathfrak{w} \leq \mathfrak{t}$ imply $\mathfrak{w} = 0$. By a general Lebesgue decomposition theorem due to Hassi, Sebestyén and de Snoo [5] (see also [7]) each nonnegative Hermitian form \mathfrak{s} can be decomposed into a sum $\mathfrak{s} = \mathfrak{s}_a + \mathfrak{s}_s$ such that \mathfrak{s}_a is absolutely continuous with respect to \mathfrak{t} and that \mathfrak{s}_s and \mathfrak{t} are mutually singular. In particular, singularity and absolute continuity are "dual" concepts in the sense that if \mathfrak{s} and \mathfrak{t} are singular with respect to each other, and \mathfrak{s} is \mathfrak{t} -absolutely continuous then $\mathfrak{s} = 0$.

Below we present an example demonstrating that a nonzero form can be simultaneously singular and pseudo-absolutely continuous with respect to another form:

Example 2.2. Let \mathcal{D} stand for the algebra of all continuous \mathbb{C} -valued functions defined on $I = [-1, 1]$. For fixed $\varphi, \psi \in \mathcal{D}$ we define the forms \mathfrak{s} and \mathfrak{t} by

$$\mathfrak{s}(\varphi, \psi) = \varphi(0)\overline{\psi(0)}, \quad \mathfrak{t}(\varphi, \psi) = \int_I \varphi \overline{\psi} d\lambda.$$

Consider now a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of \mathcal{D} satisfying $\varphi_n \geq 0$, $\text{supp } \varphi_n \subseteq [-1/n, 1/n]$ and $\int_I \varphi_n d\lambda = 1$. It is clear that

$$(\varphi | 1)_{\mathfrak{s}} = \varphi(0) = \lim_{n \rightarrow \infty} \int_I \varphi_n d\lambda = \lim_{n \rightarrow \infty} (\varphi | \varphi_n)_{\mathfrak{t}},$$

for all $\varphi \in \mathcal{D}$. Since $\mathfrak{H}_{\mathfrak{s}}$ is one-dimensional, one concludes that \mathfrak{s} is pseudo-absolutely continuous with respect to \mathfrak{t} . Nevertheless, one easily verifies that \mathfrak{s} and \mathfrak{t} are singular.

In the next theorem we present another Radon–Nikodym-type theorem which at the same time characterizes the absolute continuity, cf. also [8].

Theorem 2.3. *Let $\mathfrak{s}, \mathfrak{t}$ be nonnegative Hermitian forms on a complex vector space \mathcal{D} . The following statements are equivalent:*

- (i) \mathfrak{s} is absolutely continuous with respect to \mathfrak{t} ;
- (ii) *There is a positive selfadjoint operator S in $\mathfrak{H}_{\mathfrak{t}}$ such that $\mathcal{D} \subseteq \text{dom } S^{1/2}$ and that*

$$(2.4) \quad (S^{1/2}x | S^{1/2}y)_{\mathfrak{t}} = \mathfrak{s}(x, y), \quad \text{for all } x, y \in \mathcal{D}.$$

Proof. We are going to prove first that (ii) implies (i). Consider a $(\mathfrak{t}, \mathfrak{s})$ -sequence $(x_n)_{n \in \mathbb{N}}$ of \mathcal{D} . In the language of Hilbert spaces that means that

$$(x_n | x_n)_{\mathfrak{t}} \rightarrow 0 \quad \text{and} \quad (S^{1/2}(x_n - x_m) | S^{1/2}(x_n - x_m))_{\mathfrak{t}} \rightarrow 0,$$

by letting $n, m \rightarrow \infty$. Then, by the closability of $S^{1/2}$ we infer that

$$\mathfrak{s}(x_n, x_n) = (S^{1/2}x_n | S^{1/2}x_n)_{\mathfrak{t}} \rightarrow 0.$$

Hence \mathfrak{s} is absolutely continuous with respect to \mathfrak{t} . Conversely, by assuming \mathfrak{s} to be \mathfrak{t} -absolutely continuous, we conclude that the canonical embedding operator J of $\mathfrak{H}_{\mathfrak{t}}$ into $\mathfrak{H}_{\mathfrak{s}}$ (2.3) is closable. Hence, by a celebrated theorem of Neumann, $S := J^*J^{**}$ is a positive selfadjoint operator in $\mathfrak{H}_{\mathfrak{t}}$ such that $\mathcal{D} = \text{dom } J \subseteq \text{dom } J^{**} = \text{dom } S^{1/2}$ and that

$$(S^{1/2}x | S^{1/2}y)_{\mathfrak{t}} = (J^{**}x | J^{**}y)_{\mathfrak{s}} = (Jx | Jy)_{\mathfrak{s}} = \mathfrak{s}(x, y),$$

for all $x, y \in \mathcal{D}$. □

3. THE RADON–NIKODYM THEOREM OF ADDITIVE SET FUNCTIONS

Throughout this section T is a nonempty set and \mathcal{R} is a set-ring on T . Let β be a (real or complex valued) additive set function on \mathcal{R} . We assume in the sequel that β is bounded, that is to say,

$$M := \sup_{E \in \mathcal{R}} |\beta(E)| < \infty.$$

Then the total variation $|\beta|$ of β exists, and thus we can naturally associate a nonnegative Hermitian form \mathfrak{b} with β on the vector space \mathcal{D} of the (real or complex

valued, respectively) \mathcal{R} -simple functions by letting

$$(3.1) \quad \mathfrak{b}(\varphi, \psi) = \int_T \varphi \bar{\psi} d|\beta|, \quad \varphi, \psi \in \mathcal{D}.$$

Furthermore, we associate the Hilbert space $(\mathfrak{H}_{\mathfrak{b}}, (\cdot | \cdot)_{\mathfrak{b}})$ with \mathfrak{b} just as in the previous section. By the boundedness of β one easily verifies that the correspondence

$$\varphi \mapsto \int_T \varphi d\beta$$

defines a continuous linear functional on the dense linear manifold \mathcal{D} of $\mathfrak{H}_{\mathfrak{b}}$, namely, by the norm bound \sqrt{M} . Hence the Riesz representation theorem yields a unique representing vector $\widehat{\beta} \in \mathfrak{H}_{\mathfrak{b}}$ such that

$$(3.2) \quad \int_T \varphi d\beta = (\varphi | \widehat{\beta})_{\mathfrak{b}}, \quad \varphi \in \mathcal{D}.$$

Let be given on \mathcal{R} another (not necessarily bounded) nonnegative additive set function α . We say that β is absolutely continuous with respect to α if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\alpha(E) < \delta$ implies $|\beta|(E) < \varepsilon$ for all $E \in \mathcal{R}$, see [1, 2]. Let us denote by \mathfrak{a} and $\mathfrak{H}_{\mathfrak{a}}$ the corresponding associated Hermitian form and Hilbert space, respectively. Then we can speak about absolute continuity of the form \mathfrak{b} with respect to the form \mathfrak{a} . A natural question there arises: is there any relation between the notions of absolute continuity of additive set functions and Hermitian forms? The following lemma gives the answer, cf. also [7].

Lemma 3.1. *If the bounded nonnegative additive set function β is absolutely continuous with respect to the nonnegative additive set function α , then \mathfrak{b} is absolutely continuous with respect to \mathfrak{a} .*

Proof. A proof can be found in [7, Theorem 3.2]. However, the argument used there is based on the nontrivial apparatus of Lebesgue decomposition theory of forms (see [5, 7]), therefore we present here a new and independent proof.

Since β is α -absolutely continuous, there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ of nonnegative additive set functions such that $\beta_n \leq \beta$, $\beta_n(E) \rightarrow \beta(E)$ for all $E \in \mathcal{R}$ and that $\beta_n \leq c_n \alpha$ for some sequence $(c_n)_{n \in \mathbb{N}}$ of nonnegative numbers (see e.g. [7, Lemma 3.1]). Then, by the Riesz representation theorem there is $\widehat{\beta}_n \in \mathfrak{H}_{\mathfrak{b}}$ such that

$$\int_T \varphi d\beta_n = (\varphi | \widehat{\beta}_n)_{\mathfrak{b}}, \quad \varphi \in \mathcal{D}.$$

Similarly, for $\psi \in \mathcal{D}$ there exists $\widehat{\psi \cdot \beta}_n \in \mathfrak{H}_{\mathfrak{b}}$ such that

$$\int_T \varphi \bar{\psi} d\beta_n = (\varphi | \widehat{\psi \cdot \beta}_n)_{\mathfrak{b}}, \quad \varphi \in \mathcal{D}.$$

It is seen readily that $\|\widehat{\psi \cdot \beta}_n\|_{\mathfrak{b}} \leq \|\widehat{\psi \cdot \beta}\|_{\mathfrak{b}}$. The embedding operator J (2.3) is well defined and the following line of inequalities

$$|(\varphi | \widehat{\psi \cdot \beta}_n)_{\mathfrak{b}}| \leq \int_T |\varphi \bar{\psi}| d\beta_n \leq c_n \|\psi\|_{\mathfrak{a}} \|\varphi\|_{\mathfrak{a}}, \quad \varphi \in \mathcal{D},$$

implies that $\widehat{\psi \cdot \beta}_n \in \text{dom } J^*$ for all $\psi \in \mathcal{D}$. Since $(\varphi | \widehat{\psi \cdot \beta}_n)_{\mathfrak{b}} \rightarrow (\varphi | \widehat{\psi \cdot \beta})_{\mathfrak{b}}$ for $\varphi \in \mathcal{D}$ and since $\|\widehat{\psi \cdot \beta}_n\|_{\mathfrak{b}} \leq \|\widehat{\psi \cdot \beta}\|_{\mathfrak{b}}$ we infer that $(f | \widehat{\psi \cdot \beta}_n)_{\mathfrak{b}} \rightarrow (f | \widehat{\psi \cdot \beta})_{\mathfrak{b}}$ for all $f \in \mathfrak{H}_{\mathfrak{b}}$.

Consider now $g \in \{\text{dom } J^*\}^\perp$. Then

$$(g | \psi, \widehat{\beta})_{\mathfrak{b}} = \lim_{n \rightarrow \infty} (g | \psi, \widehat{\beta}_n)_{\mathfrak{b}} = 0, \quad \psi \in \mathcal{D}.$$

Since \mathcal{D} is dense in $\mathfrak{H}_{\mathfrak{b}}$ by definition, we may choose $(\varphi_n)_{n \in \mathbb{N}}$ of \mathcal{D} such that $\|g - \varphi_n\|_{\mathfrak{b}} \rightarrow 0$. Then for any $\psi \in \mathcal{D}$ it follows that

$$0 = (g | \psi, \widehat{\beta})_{\mathfrak{b}} = \lim_{n \rightarrow \infty} (\varphi_n | \psi, \widehat{\beta})_{\mathfrak{b}} = \lim_{n \rightarrow \infty} \int_T \varphi_n \overline{\psi} d\beta = \lim_{n \rightarrow \infty} (\varphi_n | \psi)_{\mathfrak{b}} = (g | \psi)_{\mathfrak{b}},$$

whence $g = 0$. That means that $\text{dom } J^*$ is dense in $\mathfrak{H}_{\mathfrak{b}}$ and therefore that J is closable. In other words, \mathfrak{b} is \mathfrak{a} -absolutely continuous. \square

Remark 3.2. The converse implication is also true if α is assumed to be bounded. The proof of this statement can be found in [7], cf. also [13].

We are now in position to give an extension of the Radon–Nikodym theorem due to Darst and Green [1, Theorem 1] to set functions defined on a ring of sets.

Theorem 3.3. *Let α, β be real or complex valued additive set functions on a ring of sets \mathcal{R} such that α is nonnegative and β is bounded. If β is absolutely continuous with respect to α then there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of \mathcal{R} -simple functions such that*

$$(3.3) \quad \beta(E) = \lim_{n \rightarrow \infty} \int_E \varphi_n d\alpha$$

If α is bounded too, then by letting $\beta_n(E) := \int_E \varphi_n d\alpha$ we have at the same time

$$(3.4) \quad \sup_{E \in \mathcal{R}} |\beta - \beta_n|(E) \rightarrow 0.$$

Proof. By Lemma 3.1 we conclude that \mathfrak{b} is \mathfrak{a} -absolutely continuous, and hence also \mathfrak{a} -pseudo absolutely continuous in the view of Theorem 2.1. Consequently, there exists a $(\psi_n)_{n \in \mathbb{N}}$ of \mathcal{D} such that

$$\beta(E) = (\chi_E | \widehat{\beta})_{\mathfrak{b}} = \lim_{n \rightarrow \infty} (\chi_E | \psi_n)_{\mathfrak{a}} = \lim_{n \rightarrow \infty} \int_E \overline{\psi_n} d\alpha,$$

which yields (3.3) by setting $\varphi_n := \overline{\psi_n}$. According to the proof of Theorem 2.1, $(\psi_n)_{n \in \mathbb{N}}$ may be chosen such that $\|J^* g_n - \psi_n\|_{\mathfrak{a}} \leq 1/n$ where $(g_n)_{n \in \mathbb{N}}$ is a sequence of $\text{dom } J^*$ satisfying $\|g_n - \widehat{\beta}\|_{\mathfrak{b}} \rightarrow 0$. Consequently, for any $E \in \mathcal{R}$ we infer that

$$\begin{aligned} |\beta(E) - \beta_n(E)| &\leq |(\chi_E | \widehat{\beta})_{\mathfrak{b}} - (\chi_E | g_n)_{\mathfrak{b}}| + |(\chi_E | g_n)_{\mathfrak{b}} - (\chi_E | \psi_n)_{\mathfrak{a}}| \\ &\leq \|\chi_E\|_{\mathfrak{b}} \|\widehat{\beta} - g_n\|_{\mathfrak{b}} + \|\chi_E\|_{\mathfrak{a}} \|J^* g_n - \psi_n\|_{\mathfrak{a}} \\ &\leq \sqrt{\sup_{E \in \mathcal{R}} |\beta(E)|} \cdot \|\widehat{\beta} - g_n\|_{\mathfrak{b}} + \sqrt{\sup_{E \in \mathcal{R}} \alpha(E)} \cdot \frac{1}{n}. \end{aligned}$$

This yields (3.4) if both α and β are bounded. \square

4. THE CLASSICAL RADON–NIKODYM THEOREM OF MEASURES

In this section we present the classical Radon–Nikodym theorem of measures in the following setting. Let \mathcal{R} be a σ -algebra of some subsets of T and let μ, ν be finite measures on \mathcal{R} . Similarly to the case of additive set functions, we may associate the nonnegative Hermitian forms \mathfrak{m} and \mathfrak{n} with μ and ν , respectively, on the set \mathcal{D} of simple functions. The corresponding Hilbert spaces \mathfrak{H}_{μ} and \mathfrak{H}_{ν} realize

then as the well known function spaces $\mathcal{L}^2(\mu)$ and $\mathcal{L}^2(\nu)$, respectively, thanks to the celebrated Riesz–Fischer theorem.

Recall that ν is called absolutely continuous with respect to μ if $\mu(E) = 0$ implies $\nu(E) = 0$ for any measurable set E . Our purpose in this section is to prove the classical Radon–Nikodym theorem on the representability of ν due to μ . Just as before, our method of proving is based on the canonical embedding operator J of $\mathcal{L}^2(\mu)$ to $\mathcal{L}^2(\nu)$. Our first result asserts that J is closable if and only if ν is absolutely continuity with respect to μ , cf. also [5].

Lemma 4.1. *Let μ and ν be finite measures on the measurable space (T, \mathcal{D}) , and denote the corresponding forms by \mathbf{m} and \mathbf{n} , respectively. The μ -absolute continuity of ν is equivalent to the \mathbf{m} -absolute continuity of \mathbf{n} .*

Proof. Assume that ν is absolutely continuous with respect to μ , and consider an (\mathbf{m}, \mathbf{n}) -sequence $(\varphi_n)_{n \in \mathbb{N}}$ of \mathcal{D} . That is to say,

$$\int_T |\varphi_n|^2 d\mu \rightarrow 0 \quad \text{and} \quad \int_T |\varphi_n - \varphi_m|^2 d\nu \rightarrow 0.$$

According to the Riesz–Fischer theorem we can choose a subsequence $(\varphi_{n_k})_{k \in \mathbb{N}}$ and a function $f \in \mathcal{L}^2(\nu)$ which satisfy $\varphi_{n_k} \rightarrow 0$ μ -a.e., $\varphi_{n_k} \rightarrow f$ in $\mathcal{L}^2(\nu)$ and $\varphi_{n_k} \rightarrow f$ ν -a.e. Consequently, $f = 0$ ν -a.e., and hence

$$\mathbf{n}(\varphi_n, \varphi_n) = \int_T |\varphi_n|^2 d\nu \rightarrow 0,$$

as it is claimed. That the \mathbf{m} -absolute continuity of \mathbf{n} implies the μ -absolute continuity of ν is seen readily. \square

In the language of the operator theory the μ -absolute continuity of ν means therefore that the domain of the adjoint operator J^* of J is a dense linear manifold of $\mathcal{L}^2(\nu)$. In the following lemma we are going to investigate J^* in detail:

Lemma 4.2. *Suppose that ν is absolutely continuous with respect to μ . Then we have the following on J^* :*

- (a) $f \in \text{dom } J^*$ implies $|f| \in \text{dom } J^*$.
- (b) J^* is a positive operator in the following sense: $f \in \text{dom } J^*$, $f \geq 0$ ν -a.e. implies $J^*f \geq 0$ μ -a.e.
- (c) If $f, g \geq 0$ ν -a.e., $f \in \text{dom } J^*$, $g \in \mathcal{L}^2(\nu)$ then $f \wedge g \in \text{dom } J^*$.

Proof. It is seen readily that the following inequality

$$|(|f| | \varphi)_\nu| \leq \sup_{\psi \in \mathcal{D}, |\psi| \leq |\varphi|} |(f | \psi)_\nu|$$

holds for any $f \in \mathcal{L}^2(\nu)$ and $\varphi \in \mathcal{D}$. Hence for $f \in \text{dom } J^*$ we have

$$|(|f| | \varphi)_\nu| \leq \|J^*f\|_\mu \|\varphi\|_\mu, \quad \varphi \in \mathcal{D},$$

which yields $|f| \in \text{dom } J^*$. This proves (a). Let $f \in \text{dom } J^*$, $f \geq 0$ ν -a.e. Then for each $\varphi \in \mathcal{D}$, $\varphi \geq 0$ we have

$$(J^*f | \varphi)_\mu = (f | \varphi)_\nu \geq 0.$$

Hence $J^*f \geq 0$ μ -a.e. which yields (b). Finally, if $f, g \geq 0$ ν -a.e., $f \in \text{dom } J^*$, $g \in \mathcal{L}^2(\nu)$ then

$$|(f \wedge g | \varphi)_\nu| \leq (f | |\varphi|)_\nu \leq \|J^*f\|_\mu \|\varphi\|_\mu$$

for all $\varphi \in \mathcal{D}$. Hence $f \wedge g \in \text{dom } J^*$ which proves (c). \square

We are now in position to prove the main result of the section. Our treatment below is probably not the easiest way to get the Radon–Nikodym derivative. However, it may be interesting for an operator theorist.

Theorem 4.3. *Assume that ν is absolutely continuous with respect to μ . Then there exists a μ -integrable function f such that*

$$\nu(E) = \int_E f \, d\mu, \quad E \in \mathcal{R}.$$

Proof. By Lemma 4.1 there exists $(g_n)_{n \in \mathbb{N}}$ of $\text{dom } J^*$ such that $g_n \rightarrow 1$ in $\mathcal{L}^2(\nu)$ and simultaneously, $g_n \rightarrow 1$ ν -a.e. By setting

$$f_n := 1 \wedge \left(\bigvee_{k=1}^n |g_k| \right),$$

we conclude that $f_n \leq f_{n+1}$ and that $f_n \rightarrow 1$ ν -a.e. Hence, by the B. Levi theorem $f_n \rightarrow 1$ in $\mathcal{L}^2(\nu)$ as well. At the same time, $f_n \in \text{dom } J^*$ thanks to Lemma 4.2 (a) and (c). The positivity of J^* (Lemma 4.2 (b)) yields then

$$\int_T |J^* f_n - J^* f_m| \, d\mu = (J^*(f_n - f_m) | 1)_\mu = (f_n - f_m | 1)_\nu, \quad n \geq m.$$

By letting $n, m \rightarrow \infty$ we infer that $(J^* f_n)_{n \in \mathbb{N}}$ converges to a (μ -a.e. nonnegative) function $f \in \mathcal{L}^1(\mu)$, due to the Riesz–Fischer theorem. Then f satisfies

$$\int_E f \, d\mu = \lim_{n \rightarrow \infty} \int_E J^* f_n \, d\mu = \lim_{n \rightarrow \infty} (J^* f_n | \chi_E)_\mu = \lim_{n \rightarrow \infty} (f_n | \chi_E)_\nu = \nu(E)$$

for all $E \in \mathcal{R}$. The proof is therefore complete. \square

By Lemma 4.2 one concludes that $\text{dom } J^*$ is a linear function lattice which possesses the Stone property: $1 \wedge f \in \text{dom } J^*$ for $f \in \text{dom } J^*$. Nevertheless, 1 does not belong to $\text{dom } J^*$ in general. As it turns out from Proposition 4.4 below, $1 \in \text{dom } J^*$ may happen only in the case when the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$

belongs to $\mathcal{L}^2(\mu)$ and in that case $\frac{d\nu}{d\mu} = J^* 1$.

Proposition 4.4. *Assume that ν is absolutely continuous with respect to μ . Then the following assertions are equivalent:*

- (i) *The Radon–Nikodym derivative $\frac{d\nu}{d\mu}$ belongs to $\mathcal{L}^2(\mu)$;*
- (ii) $1 \in \text{dom } J^*$;
- (iii) $\mathcal{D} \subseteq \text{dom } J^*$;
- (iv) *There is a nonnegative constant $C \geq 0$ such that*

$$\left| \int_T \varphi \, d\nu \right|^2 \leq C \int_T |\varphi|^2 \, d\mu, \quad \varphi \in \mathcal{D}.$$

In any case, $\frac{d\nu}{d\mu} = J^ 1$.*

Proof. Assume first that $\frac{d\nu}{d\mu} \in \mathcal{L}^2(\mu)$. Then for any $\varphi \in \mathcal{D}$

$$\left(\varphi \left| \frac{d\nu}{d\mu} \right.\right)_\mu = \int_T \varphi d\nu = (J\varphi | 1)_\nu.$$

This implies that $1 \in \text{dom } J^*$ and that $J^*1 = \frac{d\nu}{d\mu}$. That (ii) implies (iii) follows by Lemma 4.2. Assertion (iv) expresses precisely that $1 \in \text{dom } J^*$. Finally, if we assume (ii) then $J^*1 \geq 0$ ν -a.e. by Lemma 4.2 and

$$\int_T \varphi d\nu = (J\varphi | 1)_\nu = (\varphi | J^*1)_\mu = \int \varphi \cdot J^*1 d\mu, \quad \varphi \in \mathcal{D}.$$

Consequently, $J^*1 = \frac{d\nu}{d\mu}$ as it is claimed. \square

5. RADON–NIKODYM THEOREMS FOR REPRESENTABLE POSITIVE FUNCTIONALS ON *-ALGEBRAS

Throughout this section we fix a *-algebra \mathcal{A} and two positive functionals v, w on it. We do not assume \mathcal{A} to be unital. We shall assume v, w to be representable, that is to say, there exists a Hilbert space \mathfrak{H}_v (resp., \mathfrak{H}_w), a *-representation π_v (resp., π_w) of \mathcal{A} to $\mathcal{B}(\mathfrak{H}_v)$ (resp., $\mathcal{B}(\mathfrak{H}_w)$) and a cyclic vector ζ_v (resp., ζ_w) such that

$$v(a) = (\pi_v(a)\zeta_v | \zeta_v)_v, \quad w(a) = (\pi_w(a)\zeta_w | \zeta_w)_w, \quad a \in \mathcal{A}.$$

The cyclicity of ζ_v (resp., ζ_w) above means that $\pi_v\langle \mathcal{A} \rangle \zeta_v := \{\pi_v(a)\zeta_v | a \in \mathcal{A}\}$ (resp., $\pi_w\langle \mathcal{A} \rangle \zeta_w$) is dense in \mathfrak{H}_v (resp., in \mathfrak{H}_w).

Such a triple $(\mathfrak{H}_v, \pi_v, \zeta_v)$ can be obtained via the well-known GNS construction: set $\mathfrak{v}(a, b) = v(b^*a)$, $\mathcal{N}_{\mathfrak{v}} := \{x \in \mathcal{A} | \mathfrak{v}(x, x) = 0\}$, let $\mathfrak{H}_v := \mathfrak{H}_{\mathfrak{v}}$, and define $\pi_v(a)(b + \mathcal{N}_{\mathfrak{v}}) := ab + \mathcal{N}_{\mathfrak{v}}$. The cyclic vector is provided by Riesz representation theorem applied to the densely defined continuous linear functional $\mathfrak{H}_v \rightarrow \mathbb{C}, a + \mathcal{N}_{\mathfrak{v}} \mapsto v(a)$; see [6] for the details. The triple $(\mathfrak{H}_w, \pi_w, \zeta_w)$ can be constructed analogously.

We recall now the notion of absolute continuity in this context: w is said to be absolutely continuous with respect to v if

$$v(a_n^* a_n) \rightarrow 0 \quad \text{and} \quad w((a_n^* - a_m^*)(a_n - a_m)) \rightarrow 0, \quad n, m \rightarrow \infty,$$

imply $w(a_n^* a_n) \rightarrow 0$. We notice here that Gudder [3] called w *strongly v-absolutely continuous* in the above case. Observe immediately that w is v -absolutely continuous if and only if \mathfrak{w} is \mathfrak{v} -absolutely continuous. Equivalently, in the language of Hilbert space operators that means that the mapping

$$J : \mathfrak{H}_v \rightarrow \mathfrak{H}_w, \quad \pi_v(a)\zeta_v \mapsto \pi_w(a)\zeta_w, \quad a \in \mathcal{A},$$

is closable. The singularity of representable functionals is defined as follows (cf. also Gudder [3]): v and w are mutually singular if $p \leq v$ and $p \leq w$ imply $p = 0$ for any representable functional p on \mathcal{A} . Szűcs [10] proved that the singularity of the functionals v, w is equivalent to the singularity of their induced forms $\mathfrak{v}, \mathfrak{w}$. By using this fact and due to the general Lebesgue type decomposition theorem of Hassi, Sebestyén and de Snoo [5], Szűcs also presented a Lebesgue decomposition theorem for representable functionals, see [11]. A self-contained proof of this result can be found in [12].

Our first Radon–Nikodym-type result is a reformulation of Theorem 2.1:

Theorem 5.1. *If w is v -absolutely continuous then there exists $(a_n)_{n \in \mathbb{N}}$ of \mathcal{A} such that*

$$w(a) = \lim_{n \rightarrow \infty} v(a_n^* a), \quad a \in \mathcal{A}.$$

Moreover, the convergence is uniform on the set $\{a \in \mathcal{A} \mid w(a^ a) + v(a^* a) \leq 1\}$.*

Proof. In the view of Theorem 2.1, \mathfrak{w} is \mathfrak{v} -pseudo absolutely continuous. That means that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of \mathcal{A} such that

$$w(a) = (\pi_w(a)\zeta_w \mid \zeta_w)_w = \lim_{n \rightarrow \infty} (\pi_w(a)\zeta_w \mid \pi_v(a_n)\zeta_v)_v = \lim_{n \rightarrow \infty} v(a_n^* a),$$

for all $a \in \mathcal{A}$, as it is claimed. That the convergence on $\{a \in \mathcal{A} \mid w(a^* a) + v(a^* a) \leq 1\}$ is uniform follows immediately from Theorem 2.1. \square

Corollary 5.2. *Assume that \mathcal{A} is a Banach $*$ -algebra and let v, w be representable positive functionals such that w is v -absolutely continuous. Then there is a sequence $(a_n)_{n \in \mathbb{N}}$ such that $w_n(a) := v(a_n^* a)$, $a \in \mathcal{A}$, satisfies $\|w - w_n\| \rightarrow 0$.*

Proof. According to the proof of Theorem 2.1, $(a_n)_{n \in \mathbb{N}}$ of Theorem 5.1 may be chosen such that $\|J^* \xi_n - \pi_v(a_n)\zeta_v\|_v \leq 1/n$ where $(\xi_n)_{n \in \mathbb{N}}$ is a sequence of $\text{dom } J^* \subseteq \mathfrak{H}_w$ that satisfies $\|\xi_n - \zeta_w\|_w \rightarrow 0$. Then we have for $a \in \mathcal{A}$, $\|a\| \leq 1$ that

$$\begin{aligned} |(w - w_n)(a)| &\leq |(\pi_w(a)\zeta_w \mid \zeta_w)_w - (\pi_w(a)\zeta_w \mid \xi_n)_w| \\ &\quad + |(\pi_w(a)\zeta_w \mid \xi_n)_w - (\pi_v(a)\zeta_v \mid \pi_v(a_n)\zeta_v)_v| \\ &\leq \sup_{a \in \mathcal{A}, \|a\| \leq 1} \|\pi_w(a)\zeta_w\|_w \|\xi_n - \zeta_w\|_w \\ &\quad + \sup_{a \in \mathcal{A}, \|a\| \leq 1} \|\pi_v(a)\zeta_v\|_v \|J^* \xi_n - \pi_v(a_n)\zeta_v\|_v. \end{aligned}$$

Since each $*$ -representation of a Banach $*$ -algebra is continuous, we conclude that $\|w - w_n\| \rightarrow 0$. \square

The following result is an extension of Gudder's Radon–Nikodym theorem [3, Theorem 1] for not necessarily unital $*$ -algebras:

Theorem 5.3. *Let \mathcal{A} be a $*$ -algebra and let v, w be representable functionals on it. The following assertions are equivalent:*

- (i) *w is v -absolutely continuous;*
- (ii) *There exists a positive selfadjoint operator W on \mathfrak{H}_v such that $\pi_v(\mathcal{A})\zeta_v \subseteq \text{dom } W$ and*

$$(5.1) \quad w(b^* a) = (W\pi_v(a)\zeta_v \mid W\pi_v(b)\zeta_v)_v, \quad a, b \in \mathcal{A}.$$

Furthermore, $\pi_v(\mathcal{A})\xi \subseteq \text{dom } W$ for $\xi \in \text{dom } W$ and

$$(5.2) \quad (W\pi_v(a)\xi \mid W\eta)_v = (W\xi \mid W\pi_v(a^*)\eta)_v, \quad a \in \mathcal{A}, \xi, \eta \in \text{dom } W.$$

Proof. An application of Theorem 2.3 shows that (ii) implies (i), and that, by assuming (i), $W := (J^* J^{**})^{1/2}$ fulfills (5.1). It remains to show that W fulfills (5.2). Let $a \in \mathcal{A}$ and $\xi \in \text{dom } W = \text{dom } J^{**}$. Then there exists $(a_n)_{n \in \mathbb{N}}$ of \mathcal{A} such that $\pi_v(a_n)\zeta_v \rightarrow \xi$ and that $\pi_w(a_n)\zeta_w \rightarrow J^{**}\xi$. Hence $\pi_v(a)\pi_v(a_n)\zeta_v \rightarrow \pi_v(a)\xi$ and $J^{**}(\pi_v(a)\pi_v(a_n)\zeta_v) = \pi_w(aa_n)\zeta_w \rightarrow \pi_w(a)J^{**}\xi$. This means that $\pi_v(a)\xi \in \text{dom } J^{**}$ and that $J^{**}\pi_v(a)\xi = \pi_w(a)J^{**}\xi$. In particular we infer that

$$(5.3) \quad J^{**}\pi_v(a) \supseteq \pi_w(a)J^{**}, \quad a \in \mathcal{A}.$$

By taking adjoint,

$$(5.4) \quad \pi_v(a)J^* \subseteq J^*\pi_w(a), \quad a \in \mathcal{A}.$$

Thus an easy calculation shows that

$$(J^*J^{**}\pi_v(a)\xi \mid \eta)_v = (\xi \mid J^*J^{**}\pi_v(a^*)\eta)_v$$

holds for all $\xi, \eta \in \text{dom } J^*J^{**}$. That implies (5.2) by noticing that $\text{dom } J^*J^{**}$ is core for W . \square

Corollary 5.4. *Assume that w is v -absolute continuous and that $\zeta_v \in \text{dom } W$ (that satisfies e.g. if \mathcal{A} is unital, or if $w(a^*a) \leq Cv(a^*a)$, $a \in \mathcal{A}$, for some $C \geq 0$). Then $J^{**}\zeta_v = \zeta_w$ and*

$$(5.5) \quad w(a) = (W\pi_v(a)\zeta_v \mid W\zeta_v)_v, \quad a \in \mathcal{A}.$$

Proof. Let $a \in \mathcal{A}$. By (5.3) we conclude that

$$(J^{**}\zeta_v \mid \pi_w(a)\zeta_w)_w = (J^{**}\pi_v(a^*)\zeta_v \mid \zeta_w)_w = (\pi_w(a^*)\zeta_w \mid \zeta_w)_w = (\zeta_w \mid \pi_w(a)\zeta_w)_w,$$

hence $J^{**}\zeta_v = \zeta_w$, indeed. Consequently,

$$w(a) = (\pi_w(a)\zeta_w \mid \zeta_w)_w = (J^{**}\pi_v(a)\zeta_w \mid J^{**}\zeta_v)_w = (W\pi_v(a)\zeta_v \mid W\zeta_v)_v.$$

If \mathcal{A} is unital then $\zeta_v = 1$, and if $w \leq Cv$ then $W \in \mathcal{B}(\mathfrak{H}_v)$. In any case, $\zeta_v \in \text{dom } W$. \square

The next result generalizes [3, Theorem 1 c] of Gudder:

Proposition 5.5. *Assume that v, w are representable functionals on \mathcal{A} such that w is v -absolutely continuous. The following statements are equivalent:*

- (i) $w \leq Cv$ for some $C \geq 0$;
- (ii) $W^2 \in \text{Com}(\pi_v)$ (that is to say, $W^2 \in \mathcal{B}(\mathfrak{H}_v)$ and $W^2\pi_v(a) = \pi_v(a)W^2$ for all $a \in \mathcal{A}$).

Proof. Obviously, (i) holds if and only if $W^2 = J^{**}J^*$ is continuous. Furthermore, if (i) holds then (5.3) and (5.4) become equalities. Hence

$$W^2\pi_v(a) = J^*J^{**}\pi_v(a) = \pi_v(a)J^*J^{**} = \pi_v(a)W^2$$

for all $a \in \mathcal{A}$. \square

Corollary 5.6. *Assume that v, w are representable functionals on \mathcal{A} such that $w \leq Cv$ for some $C \geq 0$. If π_v is irreducible (that is, $\text{Com}(\pi_v) = \mathbb{C}I$) then there exists $\alpha \geq 0$ such that $w = \alpha v$.*

Proof. By Proposition 5.5, $W^2 \in \text{Com}(\pi_v)$, hence there exists $\alpha \geq 0$ such that $W^2 = \alpha I$ according to the irreducibility. Thus, by Corollary 5.4,

$$w(a) = (W\pi_v(a)\zeta_v \mid W\zeta_v)_v = (\alpha\pi_v(a)\zeta_v \mid \zeta_v)_v = \alpha v(a),$$

for all $a \in \mathcal{A}$. \square

We close our paper with characterization of absolute continuity among pure functionals on a C^* -algebra, cf. also [11]:

Corollary 5.7. *Let \mathcal{A} be a C^* -algebra such that v, w are positive functionals on \mathcal{A} . If v is pure then the following statements are equivalent:*

- (i) w is v -absolutely continuous;
- (ii) $w = \alpha v$ for some $\alpha \geq 0$.

Proof. Recall that any positive functional of a C^* -algebra is representable. Since $\pi_v\langle\mathcal{A}\rangle\zeta_v$ is obviously π_v -invariant, and since π_v irreducible, the Kadison transitivity theorem implies that $\pi_v\langle\mathcal{A}\rangle\zeta_v = \mathfrak{H}_v$. This means among others that $\text{dom } W = \mathfrak{H}_v$, and hence that $W \in \mathcal{B}(\mathfrak{H}_v)$ by the Banach closed graph theorem. Consequently, $w \leq Cv$ with some $C \geq 0$. Corollary 5.6 completes the proof. \square

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